JOURNAL OF ALGEBRA 27, 311–340 (1973)

Automorphisms of Solvable Groups

T. R. BERGER*

Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

Communicated by W. Feit

Received May 19, 1972

INTRODUCTION

In this paper we prove a representation theorem which has as a consequence:

THEOREM. *If G is a solvable group $A \leqslant \mathrm{Aut}(G)$, $C_G(A) = 1$, $(|A|, |G|) = 1$, and A is nilpotent and $\mathbf{Z}_p \wr \mathbf{Z}_p$ free for all primes p then the Fitting height of G is bounded above by the number of primes (counting multiplicities) that divide $|A|$.*

The above theorem is proved in [9]. Actually, we prove the following two results:

Let AG be a solvable group with G normal and $(|A|, |G|) = 1$. Suppose A is nilpotent and $\mathbf{Z}_p \wr \mathbf{Z}_p$ free for all primes p. Let $\mathbf{k}$ be a field of characteristic not dividing $|A|$. Assume V is a faithful irreducible $\mathbf{k}[AG]$ module. Suppose $A = A^0 > A^1 > \cdots > A^n = 1$ is a central series for A and for m, $C_V(A^m) = (0)$ but $C_V(A^{m+1}) \neq (0)$.

THEOREM A. *If $R \leqslant G$ is abelian normal in AG then there is a subgroup $D \leqslant A^m$ so that*

(a) $C_V(A^{m+1}D) = (0)$, *and*

(b) $C_R(D) \geqslant C_R(A^{m+1})$.

THEOREM B. *Suppose $R \leqslant G$ is normal in AG and has a unique maximal AG invariant subgroup $R^* \leqslant R$. Suppose $L/C_L(R/R^*)$ is an AG chief factor for some $L \leqslant G$ and $C_L(X/Y) = L$ for every AG chief factor X/Y of R^*. Finally, assume R is an r group for a prime r. Then there is a subgroup $D \leqslant A^m$ so that*

* Research partially supported by NSF grant GP 29224X.

(a) $C_V(A^{m+1}D) = (0)$, *and*

(b) $C_{R/R^*}(D) \geqslant C_{R/R^*}(A^{m+1})$, *and*

(c) $C_{G/F(G)}(D) \geqslant C_{G/F(G)}(A^{m+1})$.

The first general theorem of this type appears in [17, (4.1)] proved by E. Shult. This theorem was for abelian A and excluded certain numbers; among them were the Mersenne and Fermat primes. The extension of Shult's theorem to a class 2 odd p group appears in [2, (VI.1)] again with the prime exclusions. Still with conditions upon primes these results were extended in [II, (11.1)] to arbitrary class odd p groups. By exploiting the same technique the most general theorem of this type was announced in [10] still with conditions upon divisors of $|AG|$.

The present theorems are an outgrowth of these earlier ones. Basically we are interested in how A transmits its action through G to V. Assume for simplicity that $\mathbf{k}$ is algebraically closed. Since G is a normal Hall complement to A it is not difficult to see that there is a subgroup A_1G_1 ; $A_1 \leqslant A$, $G_1 \leqslant G$; and a primitive $\mathbf{k}[A_1G_1]$ module U so that $U|^{AG} \simeq V$. We set $W = U|^{A_1G}$. Now the question breaks naturally into three parts.

(1) For the primitive A_1G_1 module U, how does A_1 transmit its action through G_1 to U?

(2) Given the answer to (1), what effect does the induction to W have upon the action of A_1 upon W?

(3) Given the action upon W, what does the induction from A_1 to A do for the action upon V?

We will work our way backward through these questions. This is how the proofs of A and B proceed using induction to obtain answers to prior cases. First note that $V|_A \simeq W|^{AG}|_A \simeq W|_{A_1}|^A$. In particular, G does not affect the action of A on V in any essential way above W. Thus to prove a theorem we need only have a conclusion that will "survive" this induction from W to V. In Shult's theorem, A was abelian. The crucial point here was that all subgroups of A are normal. What this said is that, for almost any subgroup he chose to use in his conclusion, this choice would survive the induction from W to A. In the next steps beyond Shult's theorem, A was not necessarily abelian. But normality could still be preserved by choosing a group from some central section A^m/A^{m+1} of A. This is what was done earlier for A/A' and this is what is done here for A^m/A^{m+1}. So this last step (3) does not affect the outcome essentially. It requires only that we be somewhat cautious in our choice of conclusions.

Now consider the second question. For simplicity assume that A_1G_1 is maximal in A_1G. That is, for some A_1G chief factor H/K of G we have

$G_1H = G$ and $G_1 \cap H = K$. For $x_1 = 1, \ldots, x_t$ coset representatives of K in H we have

$$W|_{A_1} \simeq U|^{A_1G}|_{A_1} \simeq \sum x_i \otimes U|_{(A_1G_1)^{x_i} \cap A_1}|^{A},$$

where the sum is over some of the x_i's. Suffice it to say that we may choose x_i so that $C_{A_1}(x_i) = (A_1G_1)^{x_i} \cap A_1$. Therefore the action of A_1 upon H/K as a permutation group of the elements of H/K is of crucial importance. That is, A_1 will permute the $x_i \otimes U$ just as it permutes the x_iK. Here is the first point at which the action of A upon V is affected directly by the action of A upon G. The effect occurs due to the permutation action of A_1G_1 upon the chief factor H/K. Suppose $C_{A_1}(H/K) = A_0$. If A_1 has a regular orbit upon H/K then, for some x_i, $A_0 = C_{A_1}(x_i) = (A_1G_1)^{x_i} \cap A_1$ and so $W|_{A_1}$ has an A_1 direct summand isomorphic to $U|_{A_0}|^A$. Thus the "essential" action of A_1 on W is determined by the action of A_0 on U. But A_0 centralizes H/K and so the induction $U|^{A_1G}$ does not affect the action of A_0! So it appears that an answer to (2) requires we investigate the following question:

(I) When does A have a regular orbit on the elements of V?

Here we are now viewing V as H/K and $A_1G/C_{A_1G}(H/K)$ as AG. An answer to (I) will give us a handle on (2).

Finally, consider the primitive module U. For simplicity again we assume A_1G_1 is faithful on U. Now the Fitting subgroup is a central product of various extra special groups with cyclic groups which lie in $Z(A_1G_1)$. We may choose each extra special factor R so that $R/Z(R)$ is an A_1G_1 chief factor.

Look for a moment at $\mathrm{SL}(2, 3) = CQ$, where C is cyclic of order 3 and Q is quaternion. It is well known that if χ is a complex irreducible faithful character then $\chi(1) = 2$ and $\chi|_C$ is a sum of two distinct linear characters of C. Now χ may be so chosen that neither of these characters is the identity. That is, if U is a module affording χ then C is fixed point free both on $Q/Z(Q)$ and U. In this case we see that $3 = 2^1 + 1$ where $|Q| = 2^{2.1+1}$.

It was at this point that Shult, and also the later theorems, introduced hypotheses on primes to get rid of the situation just described. In fact, there is no reason that we might not have $A_1 = C$, $G_1 = R = Q$. Thus an answer to (3) requires an answer to the following question:

(II) If $\mathbf{k}$ is algebraically closed, V is primitive, and $R \leqslant G$ is extra special normal in AG, then how does R affect the action of A upon V? In particular, if A is faithful upon R, when does $V|_A$ contain a regular A direct summand?

If, in answer to the last question, $V|_A$ always contains a regular A direct summand then for A_1G_1 on U we will have $C_{A_1}(R)$ acting fixed point freely upon U if A_1 acts that way. This situation was guaranteed by prime hypotheses in earlier theorems.

So it appears that answers to (I) and (II) are fundamental to a proof of a theorem like B. It is striking that the study of (I) and (II) predates even the earliest theorems of Bauman [1], Hoffman [16], and Shult [17]. These questions were first considered by Hall and Higman [15] in their landmark paper. As a matter of fact, Shult's methods come directly from Hall–Higman at crucial points. In fact, Shult points out that his preliminary results are just a "relatively prime" Hall–Higman theorem.

These observations became clearer in the author's paper [2]. There, Section III was devoted to an answer to (I) for odd class 2 p groups. In Sections IV and V answers to (II) were given also for odd class two p groups. In [11] entire effort was spent upon (I) and (II) passing off rather too lightly that the necessary representation theorems were obvious. It was in this paper that the beginnings of a general method were exploited.

Thus the questions (I) and (II) become important in their own light. In fact, they are of interest both in the "relatively prime" setting of Shult and in the "modular" setting of Hall–Higman. A generalized form for (I) and (II) is the major consideration in the sequence "Hall–Higman type theorems" by the author. The results of import here are given in [VI]. They are (for the hypothesis stated at the outset):

THEOREM I. *A will always have a regular orbit on the elements of V.*

THEOREM II. *If $\mathbf{k}$ is algebraically closed, V primitive, A faithful on $R/Z(R)$, and $C_G(R/Z(R)) < G$, then $V|_A$ contains a regular A direct summand.*

It should be noted that $\mathbf{Z}_p \wr \mathbf{Z}_p$ freeness is essential in both Theorem I and Theorem II. So Theorem I supplies an answer to (I) and allows us to complete the answer to (2). And Theorem II supplies the answer to (II) which allows us to complete (3).

Of course this oversimplifies the actual situation. But the essence is still there. There is considerable jockeying for the positions described in (1), (2), and (3) above. For example, the questions (2) and (3) are investigated in depth above $F(G)$ in Section IV. Digging out the "support subgroup" of Theorem B is done in Section III. Properties of certain p groups are investigated in Section II. Of course there is a preliminary Section I. The main Theorem B occurs in Section V. Question (1) is disposed of at (5.6). After much hammering, question (2) is beaten down finally in (5.14). Then (3) is quickly disposed of in (5.15) and (5.16).

Section VI is an anticlimax where Theorem A is observed.

I. Preliminary Results

The notation here follows [I–VI] and earlier papers of the author. Familiarity with [2] and [17] might help. Much of what has happened since [2] is an elaborate orchestration of germs contained in [17, (4.1)]. In what follows certain results are cited directly from standard texts [12, 14]. Further, certain standard theorems go without name or citation (Clifford's theorems, Mackey's theorems, etc.). Thus, since familiarity is a relative thing, the reader may find an obvious result quoted and a not so obvious result used without mention. Results are also directly quoted from [I–VI].

Of course all groups are finite and solvable whether mentioned or not. This hypothesis is not always needed, but we need not split hairs.

(1.1) *Suppose G is a group and $N_1, \ldots, N_t$ are normal subgroups such that $\bigcap N_i = N_0$. Let $\bar{G}_i = G/N_i$. Form $\tilde{G} = \bar{G}_1 \times \cdots \times \bar{G}_t$. The map ϕ: $G \to \tilde{G}$ given by $\phi(x) = (xN_1, \ldots, xN_t)$ is a homomorphism of G into $\tilde{G}$ with kernel N_0. If $L \leqslant G$ set $\bar{L}_i = LN_i/N_i$ and $\tilde{L} = \bar{L}_1 \times \cdots \times \bar{L}_t$, then $\phi(L) \leqslant \tilde{L}$.*

This proposition is an elementary computation.

(1.2) *Suppose G is a group and $N_1, \ldots, N_t$ are normal subgroups such that $\bigcap N_i = 1$. Let $F_i/N_i = F(G/N_i)$. Then $\bigcap F_i = F(G)$.*

Clearly $F_0 = \bigcap F_i \geqslant F(G)$. Also note that $F_0 \leqslant F_i$ for every i. Now $F_0N_i/N_i \leqslant F(G/N_i)$ is nilpotent and normal in G/N_i. Thus

$$\phi(F_0) \leqslant F(\tilde{G}) = F(\bar{G}_1) \times \cdots F(\bar{G}_t).$$

So $\phi(F_0) \leqslant F(\tilde{G}) \cap \phi(G) \leqslant F(\phi(G))$. Since ϕ is an isomorphism we get

$$F_0 \leqslant F(G).$$

Therefore $F_0 = F(G)$.

(1.3) *Suppose A is a relatively prime operator group on G. If $Y \Delta X \leqslant G$ are A invariant subgroups and A fixes the coset $xY \in X/Y$, then A fixes some $xy \in xY$. Thus $C_{X/Y}(A) = C_X(A)Y/Y$.*

A proof is given in [13]. We now derive corollaries of this proposition.

(1.4) *Suppose A is a relatively prime operator group on solvable G. If N is A invariant and normal in G and A centralizes every A chief factor of G/N then A centralizes G/N.*

Since A operates upon G/N we may factor N out and assume $N = 1$. Then (1.3) will complete the proof when $N \neq 1$. We assume $N = 1$ and show A centralizes G. Let M be minimal A invariant and normal in G.

Then M is an A chief factor of G so A centralizes M. By induction upon $|G|$, A centralizes G/M. So by (1.3) A centralizes G, since it centralizes G/M and M.

(1.5) [13] *Suppose A is a relatively prime operator group on G. Then $[G, A] = [G, A, A]$ is normal in G.*

(1.6) *Suppose A is a relatively prime operator group on G. Assume $G_1 \leqslant G$ is A invariant, $G_1 C_G(A) = G$, and $G_0 = \bigcap_{x \in G} G_1$. Then G_0 is A invariant and $G_0 C_G(A) = G$.*

Form the semidirect product AG. Now $G_1^{x\alpha} = G_1^{\alpha^{-1}x\alpha}$, where $x \in G$, $\alpha \in A$. Thus G_0 is A invariant and normal in G. Let $\mathscr{T} = \{x_1 = 1, x_2, \ldots, x_t\} \subseteq C_G(A)$ be a transversal of G_1 in G. Then $\bigcap_{x \in G} G_1^x = \bigcap_{x \in \mathscr{T}} G_1^x = G_0$. Thus for $G_i = G_1^{x_i}$ we have $G_i C_G(A) = G$ and G_i is A invariant.

Let $y \in G$. For some $x \in \mathscr{T}$ and $w \in G_1$ we have $y = xw$. If $\alpha \in A$, then $[\alpha, y] = [\alpha, xw] = [\alpha, w][\alpha, x][\alpha, x, w] = [\alpha, w] = w^{-\alpha} w \in G_1$. Therefore $[G, A] \leqslant G_1$. A similar argument shows that $[G, A] \leqslant G_i$ for all i. In fact, $G_i = G_1^{x_i} \geqslant [G, A]^{x_i} = [G, A]$. But then $[G, A] \leqslant G_0$. Now A centralizes G/G_0 so by (1.3) $G = G_0 C_G(A)$.

(1.7) *Assume A is a relatively prime operator group on G. Suppose $A^* \leqslant A$ and $N_1, \ldots, N_t$ are A invariant normal subgroups of G. If $N_i C_G(A^*) = G$ and $N = \bigcap N_i$ then $NC_G(A^*) = G$.*

By (1.3) we may factor N out of G and assume it is 1. Consider the map of (1.1). Note that it is an A isomorphism of G. Since A^* centralizes each $\bar{G}_i$, $A^* \leqslant A$ centralizes $\tilde{G}$. Thus A^* centralizes G.

(1.8) *Assume that A is a relatively prime operator group on G. Suppose $N_1, \ldots, N_t$ are A invariant normal subgroups of G and $A_1, A_2 \leqslant A$. Let $\bar{G}_i = G/N_i$. Assume that*

$$C_{\bar{G}_i/F(\bar{G}_i)}(A_1) \geqslant C_{\bar{G}_i/F(\bar{G}_i)}(A_2)$$

and $\bigcap N_i = 1$. Then

$$C_{G/F(G)}(A_1) \geqslant C_{G/F(G)}(A_2).$$

Let $F_i/N_i = F(\bar{G}_i)$. Then by (1.2) $\bigcap F_i = F(G)$. Now by (1.3) for $A^* \leqslant A$ we have $C_{\bar{G}_i/F(\bar{G}_i)}(A^*) = C_{\bar{G}_i}(A^*)F(\bar{G}_i)/F(\bar{G}_i)$. And $C_{\bar{G}_i}(A^*) = C_G(A^*)N_i/N_i$. Thus

$$C_{\bar{G}_i/F(\bar{G}_i)}(A^*) = (C_G(A^*)F_i/N_i)/(F_i/N_i).$$

In particular,

$$C_G(A_1)F_i \geqslant C_G(A_2)F_i .$$

But by (1.7)

$$\bigcap C_G(A^*)F_i = C_G(A^*) \bigcap F_i$$
$$= C_G(A^*) F(G)$$

by (1.2). So

$$C_G(A_1) F(G) = \bigcap C_G(A_1)F_i \geqslant \bigcap C_G(A_2)F_i = C_G(A_2) F(G).$$

The conclusion now follows from (1.3).

We now state a few other results.

(1.9) [14, (5.4.6)] *Let C be a normal extra special subgroup of the p-group P such that P is trivial on $C/Z(C)$ then $P = C_P(C)C$.*

(1.10) *Assume H is a group and $\mathbf{k}$ is a field of characteristic c where $c \nmid |H|$. Suppose V is a $\mathbf{k}[H]$ module. If $\hat{\mathbf{k}} > \mathbf{k}$ is an algebraically closed field then, for $\hat{V} = \hat{\mathbf{k}} \otimes_{\mathbf{k}} V$,*

$$C_{\hat{V}}(H) = \hat{\mathbf{k}} \otimes_{\mathbf{k}} C_V(H).$$

Let $\epsilon \in \mathbf{k}$ be a primitive $|AG|^{\text{th}}$ root of unity. Set $\mathbf{k}^* = \mathbf{k}(\epsilon)$. Let $V^* = \mathbf{k}^* \otimes_{\mathbf{k}} V$. Now $\hat{V} = \hat{\mathbf{k}} \otimes_{\mathbf{k}^*} V^*$. We first prove that

$$C_{\hat{V}}(H) = \hat{\mathbf{k}} \otimes_{\mathbf{k}^*} C_V{}^*(H).$$

By [12, (41.1)] $V^* = V_1{}^* \dotplus \cdots \dotplus V_s{}^*$, where the $V_i{}^*$ are absolutely irreducible $\mathbf{k}^*[H]$ modules. By [12, (29.21)] $\hat{\mathbf{k}} \otimes_{\mathbf{k}^*} V_i{}^* = \hat{V}_i$ is also absolutely irreducible. Thus

$$\hat{V} = \hat{V}_1 \dotplus \cdots \dotplus \hat{V}_s .$$

Number the V_i so that $[H, V_i] = V_i$ for $i = 1,..., t$ and $[H, V_i] = (0)$ for $i = t + 1,..., s$. Thus $\hat{V} = (\hat{V}_1 \dotplus \cdots \dotplus V_t) \dotplus (\hat{V}_{t+1} \dotplus \cdots \dotplus \hat{V}_s)$. Note that $C_{\hat{V}}(H) = \hat{V}_{t+1} \dotplus \cdots \dotplus \hat{V}_s$. Also note that $[H, V_i{}^*] = V_i{}^*$, $i = 1,..., t$, and $[H, V_i{}^*] = (0)$ for $i = t + 1,..., s$. So $C_{V^*}(H) = V_{t+1}^* \dotplus \cdots \dotplus V_s{}^*$. Therefore

$$C_{\hat{V}}(H) = \hat{\mathbf{k}} \otimes_{\mathbf{k}^*} C_{V^*}(H).$$

So to complete the proof we need only show that

$$C_{V^*}(H) = \mathbf{k}^* \otimes_{\mathbf{k}} C_V(H).$$

Let $v_1 ,..., v_s$ be a $\mathbf{k}$ basis for $[H, V]$ and $u_1 ,..., u_t$ a $\mathbf{k}$ basis for $C_V(H)$. Then $\hat{v}_i = 1 \otimes v_i$ and $\hat{u}_i = 1 \otimes u_i$ form a $\mathbf{k}^*$ basis of V^*. Suppose $\hat{v} \in C_V{}^*(H)$. Then we may assume that $\hat{v} = \sum \beta_j \hat{v}_j$. Now since $[\mathbf{k}^*: \mathbf{k}]$ is finite we may choose a basis and a familiar argument shows that all $\beta_j = 0$, completing the proof.

(1.11) *Assume A is nilpotent, $A^* \lhd A$, $A_0 \leqslant A$. Let $\mathbf{k}$ be a field of characteristic c where $c \nmid |A|$. Let U be $\mathbf{k}[A_0]$ module and $V = U|^A$. Then $C_V(A^*) = (0)$ if and only if $C_U(A_0 \cap A^*) = (0)$.*

By (1.10) we may assume that $\mathbf{k}$ is algebraically closed just by tensoring with an algebraic closure $\hat{\mathbf{k}}$ of $\mathbf{k}$. Now the characteristic of $\mathbf{k}$ does not divide $|A|$. So the representation theory of A is not modular. So the Brauer characters of A and its subgroups are ordinary characters. In particular we may assume $\mathbf{k} = \mathbf{C}$, the complex field.

Let χ be the character of U. Let $\mathscr{T}$ be a set of coset representatives for A_0 in A. Then

$$\begin{aligned}(\chi|^A|_{A^*}, 1_{A^*}) &= \Big(\sum_{x\in\mathscr{T}} \chi^x|_{A_0^x\cap A^*}|^{A^*}, 1_A{}^*\Big)\\ &= \sum_{x\in\mathscr{T}} (\chi^x|_{A_0^x\cap A^*}, 1_{A_0^x\cap A^*})\\ &= \sum_{x\in\mathscr{T}} ((\chi|_{A_0\cap A^*})^x, (1_{A_0\cap A^*})^x)\\ &= [A: A_0](\chi|_{A_0\cap A^*}, 1_{A_0\cap A^*}).\end{aligned}$$

Thus $\chi|^A$ contains a character trivial on A^* if and only if χ contains a character trivial on $A_0 \cap A^*$. This proves (1.11).

Remark. For $A^* = A$ this says $C_V(A) = (0)$ if and only if $C_U(A_0) = (0)$.

II. Type e p Groups

Let p be a prime and P a p group. We say P is of *type e* if P is the central product

$$P \simeq EC$$

of an extra special group E and a group C such that

(i) C is cyclic, or

(ii) $|C| \geqslant 16$ and C is dihedral, semidihedral, or generalized quaternion.

If (i) always holds we call P of *type e′*.

We have the following familiar theorem of P. Hall, to which an extra part has been added.

(2.1) [14, (5.4.9)] *If P is a p group and every characteristic abelian subgroup of P is cyclic, then P is of type e. If, in addition, p is odd, then $P = \Omega(P)\, Z(P)$ where $\Omega(P) = \{x \mid x^p = 1\}$ is extra special whenever P is non-abelian.*

Assume P is a non-abelian odd p group of type e. We may write $P \simeq EZ(P)$, where E is extra special and $Z(P)$ is cyclic. If $\Omega(E) = E$, we are through. So assume $\Omega(E) < E$. Now p is odd and class $E = 2$ so $\Omega(E)$ is a group. The map $x \rightarrow x^p$ now must be a homomorphism of $P \rightarrow P$ with kernel $\Omega(P) \geqslant \Omega(E)$. Since $\Omega(P)$ is characteristic, $Z(\Omega(P))$ is also characteristic. But $Z(\Omega(E)) \leqslant Z(\Omega(P))$. So $Z(\Omega(E))$ must be cyclic.

On the other hand, $x \rightarrow x^p$ maps E onto $Z(E)$ since $\Omega(E) < E$. So $\Omega(E) \geqslant Z(E)$ and has order $|E|/p$. Thus $|Z(\Omega(E))| = p^2$ and $Z(\Omega(E))$ is elementary abelian. So $Z(\Omega(E))$ is non-cyclic. This contradiction proves that $\Omega(P) = \Omega(E)\,\Omega(C) = E$. So (2.1) is complete.

(2.2) *If $p = 2$ and P is of type e and exponent $f \geqslant 8$, then P contains a characteristic cyclic subgroup D of exponent $f/2$. Further, $D \leqslant Z(P)$ if and only if P is of type e'.*

Write $P = EC$ where E is extra special and C is cyclic; or dihedral, semidihedral, or generalized quaternion of order $|C| \geqslant 16$. If $w \in P$ then $w = xy$ where $x \in E$ and $y \in C$. Now E has exponent 4 and $[x, y] = 1$ so that the exponent of P is the exponent of C. If C is cyclic then $Z(P) = C$ so C^2 is the desired subgroup D. Note there that P is of type e'.

Assume C is non-cyclic. Then C contains a unique maximal cyclic subgroup B of exponent f. Let $w \in P$ have order f. Then $w = xy$ where $x \in E$ and $y \in B$. Also $x^2 \in Z(E) \leqslant B^2$ and x^2 has order $\leqslant 2$. Now $f \geqslant 8$ so y^2 has order $\geqslant 4$ thus $\langle y^2 \rangle = B^2$ and $\langle w^2 \rangle = B^2$.

For any element $w = xy$, $x^2 \in B^2$ and $y^2 \in D(C) = B^2$, the Frattini subgroup of C. Thus $B^2 = \langle x^2 \mid x \in P \rangle$ is the desired cyclic subgroup D. Note that if C is non-cyclic then $D \nleqslant Z(C)$ so $D \nleqslant Z(P)$.

(2.3) [14, (5.5.2)] *If P is an extra special 2 group then $P \simeq Q \cdots Q$ (r copies) or $P \simeq Q \cdots QD$ ($r - 1$ copies of Q) where Q is quaternion and D is dihedral of order 8.*

(2.4) *If P is a 2 group of type e' then P is quaternion of order 8 or P is generated by $Z(P)$ and its elements of order 2.*

Assume P is not quaternion. Now $P = EZ(P)$. Let D be dihedral of order 8 and Q be quaternion. If $E \simeq D$ then E is generated by its elements of order 2 and (2.4) holds. If $E \simeq Q$ then $|Z(P)| \geqslant 4$. Let $x, y \in E$ generate E. Let $z \in Z(P)$ have order 4. Now $xz, yz, Z(P)$ generate P and $(xz)^2 = (yz)^2 = 1$.

Now assume that $E \simeq QQ \cdots Q$ (r copies) or $E \simeq Q \cdots QD$ ($r - 1$ copies of Q) where $r - 1 \geqslant 1$. We use induction upon r for $r \geqslant 2$ to show that the elements of order 2 in E generate E. Suppose $r = 2$. Assume, first, that

$$E \simeq QQ = Q_1 Q_2 \,.$$

Choose $x_i, y_i \in Q_i$, $i = 1, 2$ to generate Q_i. Now x_1y_1, x_1y_2, x_2y_1, x_2y_2 each has order 2 and generate E.

Second, assume that

$$E \simeq QD.$$

Choose $x, y \in Q$ to generate Q. Choose $a, b \in D$ of order 2 to generate D. Let $c \in D$ have order 4. Then xc, yc, a, b all have order 2 and generate E.

Assume $r > 2$ and

$$E^* \simeq Q \cdots Q \qquad (r - 1 \text{ copies})$$

is generated by $a_1, \ldots, a_n$ all of order 2. Let $c \in E^*$ have order 4. Suppose

$$E \simeq E^*D.$$

Choose $a, b \in D$ of order 2 generating D. Then $a_1, \ldots, a_n$, a, b all have order two and generate E.

Suppose

$$E \simeq E^*Q.$$

Choose $a, b \in Q$ to generate Q. Then $a_1, \ldots, a_n$, ac, bc all have order 2 and generate E. So by induction upon r, when $r \geqslant 2$, E is generated by its elements of order 2.

Choose $a_1, \ldots, a_n$ of order 2 in E to generate E. These with $Z(P)$ clearly generate P.

(2.5) *Let P be a p group of type e'. Suppose M is a p group acting as automorphisms of P centralizing $P/Z(P)$ and $Z(P)$. Then M centralizes P/P'.*

Assume p is odd. Then $\Omega(P)\, Z(P) = P$ and $P' = \Omega(P) \cap Z(P)$. Thus

$$P/P' = \Omega(P)/P' \times Z(P)/P'$$

is an M invariant decomposition. Clearly M centralizes $Z(P)/P'$. But

$$P/Z(P) = \Omega(P)\, Z(P)/Z(P) \simeq \Omega(P)/\Omega(P) \cap Z(P) = \Omega(P)/P'$$

is an M isomorphism so that M centralizes P/P'.

Assume $p = 2$. Certainly M centralizes $Z(P)/P'$. If P is quaternion of order 8 then $P/P' = P/Z(P)$ and we are done. So assume P is not quaternion of order 8. Then P is generated by its elements of order 2 and $Z(P)$. Let $a_1, \ldots, a_n$ be of order 2 and $\langle a_0 \rangle = Z(P)$ so the a_i's generate P. Let $x \in M$. Now $a_i^x = a_i z_i$ for some $z_i \in Z(P)$ since M centralizes $P/Z(P)$. But $(a_i^x)^2 = a_i^2 z^2 = z^2 = 1$ for $i > 0$; and $z^2 = 1$ obviously for $i = 0$. So $z \in P'$. Thus M centralizes

$$P/P' = \langle a_0, \ldots, a_n \rangle / P'.$$

(2.6) *Let P be a non-abelian p group of type e'. Suppose M acts as automorphisms of P irreducible on $P/Z(P)$ and centralizing $Z(P)$. If M is solvable, then $P = NZ(P)$ where N is extra special and M invariant.*

If P is extra special, then take $N = P$. So we may assume $|Z(P)| > p$. Let $\bar{M} = M/C_M(P)$. Now $O_p(\bar{M})$ centralizes P/P' by (2.5) since it is trivial on $P/Z(P)$ and $Z(P)$. Let $\bar{T}/O_p(\bar{M})$ be minimal normal in $\bar{M}/O_p(\bar{M})$. Then $\bar{T}$ is fixed point free on $P/Z(P)$ and trivial on $Z(P)$. Now

$$P/P' = [\bar{T}, P/P'] \times C_{P/P'}(\bar{T}).$$

But $C_{P/P'}(\bar{T}) = Z(P)/P'$ and $[\bar{T}, P/P'] \simeq P/Z(P)$. Now $[\bar{T}, P/P']$ is M invariant. Its inverse image N in P will be extra special and M invariant with $NZ(P) = P$.

(2.7) *Assume that P is non-abelian of type e'. Assume P is a normal subgroup of M and every abelian subgroup N of P which is normal in M is in $Z(M)$. Then $P/Z(P)$ is a completely reducible M module. Further, if $H/Z(P) \neq 1$ is irreducible as an M module, then $Z(H) = Z(P)$ and H is of type e'.*

Note that $g(xZ(P), yZ(P)) = [x, y] \in P' \simeq \mathrm{GF}(p)^+ = \mathbf{K}$ induces a non-singular symplectic form on $P/Z(P)$. (See [IV].) Since $Z(P) \leqslant Z(M)$, M fixes this form. Let $H/Z(P) \neq 1$ be an irreducible M submodule of $P/Z(P)$. Since $H/Z(P)$ is irreducible, it is either totally isotropic or non-singular for g. If it is totally isotropic then H is abelian and normal in M. Thus $H \leqslant Z(M) \cap P \leqslant Z(P)$, contradicting the fact that $H/Z(P) \neq 1$. Thus H is non-abelian and normal in M.

Let N be characteristic and abelian in H. Then $N \leqslant P$ and $N \triangle M$ so that $N \leqslant Z(M) \cap P \leqslant Z(P)$ is cyclic. By (2.1) and (2.2) H is of type e'. Since $H/Z(P)$ is non-singular for g, $Z(H) = Z(P)$.

Let $K/Z(P) = \{xZ(P) \mid g(xZ(P), yZ(P)) = 0 \text{ all } y \in H\}$. That is, $K/Z(P)$ is the g orthogonal complement of $H/Z(P)$. Now g is non-singular on $P/Z(P)$ and $H/Z(P)$ so that $P/Z(P) = H/Z(P) \dotplus K/Z(P)$. Since M fixes the form g, this is an M decomposition. Thus $P/Z(P)$ is a completely reducible M module.

III. Support Subgroups

Assume G is a solvable group with normal subgroup L. We call H an L *support subgroup* of G provided:

(1) H is a normal subgroup of G;

(2) H is an r subgroup for some prime r;

(3) H contains a unique maximal G invariant subgroup $H^* < H$;

(4) $L/C_L(H/H^*)$ is a non-trivial G chief factor and $C_L(X/Y) = L$ for all G chief factors of H^*.

We denote the section H/H^* by $\hat{H}$. First we prove a few properties of L support subgroups.

(3.1) *Assume $N \triangle G$ and $N \ngeqslant H$. If H is an L support subgroup of G, then HN/N is an LN/N support subgroup of G/N with $\hat{H} \simeq (HN/N)/(H^*N/N)$ as G modules.*

Let $\phi: G \to G/N = \bar{G}$ be the natural map. If $X \leqslant G$ set $\phi(X) = \bar{X}$. Now ϕ is clearly a G homomorphism. Note that $N \cap H < H$ and $N \cap H$ is normal in G so $N \cap H \leqslant H^*$. This means $\bar{H}^*$ is the unique maximal $\bar{G}$ invariant subgroup of $\bar{H}$. As G modules $\hat{H} \simeq \hat{\bar{H}}$. Thus $N \leqslant C_G(\hat{H})$. Now $N \cap L \leqslant C_L(\hat{H})$ so that $L/C_L(\hat{H}) \simeq \bar{L}/C_{\bar{L}}(\bar{H})$ is a G isomorphism. Further, any chief G factor of $\bar{H}^*$ comes from a chief G factor of H^*. Thus, for such a section X, $C_{\bar{L}}(X) = \bar{L}$. This completes the proof.

(3.2) *If H is an L support subgroup of G, then H/H' has prime exponent r.*

Since $H' \triangle G$ we see that $H^* \geqslant H'$. By the previous lemma we may replace G by G/N where $N = H'$, H by H/N, and L by LN/N. Thus we may assume H is abelian. Let $\Omega = \Omega_1(H) = \{x \in H \mid x^r = 1\}$. Now $\Omega \triangle G$. If $\Omega \neq H$ then $\Omega \leqslant H^*$. Assume $\Omega \neq H$. Consider the map $\phi: x \to x^r$ for $x \in H$. This is clearly a G homomorphism of H. The kernel of ϕ is Ω. Let $H_0 = \phi(H)$ and $H_0^* = \phi(H^*)$. Now $H_0 \triangle G$ so that $H_0 \leqslant H^*$. Since ϕ is a G homomorphism and $\Omega \leqslant H^*$, $H/H^* \simeq \phi(H)/\phi(H^*) = H_0/H_0^*$ is a G isomorphism. But then $C_L(H_0/H_0^*) = L$ since H_0/H_0^* is a G chief factor of H^*. This contradicts the fact that $C_L(H_0/H_0^*) = C_L(H/H^*) < L$. Thus $\Omega = H$ and (3.2) holds.

(3.3) *If H is an L support subgroup, then H has class $\leqslant 2$.*

Assume class $H > 2$. Let $H^0 = H$ and $H^i = [H, H^{i-1}]$. Note that $H^i \triangle G$. Also $H^1 \leqslant H^*$. Since H has class at least 3, $H^0 > H^1 > H^2 > H^3$. Set $H^3 = N$ and apply (3.1) treating G/N, H/N, and LN/N. Thus we may assume $H^3 = 1$. For each $x \in H^1$ we define $\phi_x(y) = [x, y]$. This is a map of H into H^2. Now $[x, w] = 1$ for all $w \in H^1$ so that we have

$$\phi_x(yz) = [x, yz] = [x, y][x, z] = \phi_x(y)\,\phi_x(z).$$

Thus ϕ_x is a homomorphism of H into H^2 with kernel containing H^1.

Choose L_0 minimal in L such that $L_0C_L(\hat{H}) = L$. Since $L/C_L(\hat{H})$ is an elementary abelian s group for some prime $s \neq r$, L_0 is an s group and the Frattini subgroup $D(L_0) = L_0 \cap C_L(\hat{H})$.

Let $H^* = H_1 > \cdots > H_t = 1$ be a chief G series of H^*. Since $H_i/H_{i+1} = \bar{H}_i$ is a chief G factor of H^*, $C_L(\bar{H}_i) = L$. That is, L_0 stabilizes the chain $H^* = H_1 > \cdots > H_t = 1$. But L_0 is an r' group so L_0 centralizes H^*. In particular L_0 centralizes H^1. Thus, for each $x \in H^1$, ϕ_x is an L_0 homomorphism.

Now L_0 acts as $L_0/D(L_0) \simeq L/C_L(H)$ upon $\hat{H}$. That is, L_0 acts fixed point freely. Let T/H^* be a non-trivial irreducible L_0 submodule of $\hat{H}$. Since $H^2 = \langle \phi_x(H) \mid x \in H^1 \rangle$ we may choose $x \in H^1$ so that $(\ker \phi_x) H^* < T$. Since ϕ_x is an L_0 homomorphism and T/H^* is L_0 irreducible we have $T/H^* \simeq \phi_x(T)/\phi_x(H^*)$ is an L_0 isomorphism. But the latter is a section of $H^2 \leqslant H^*$ so L_0 must be trivial on it. This contradiction proves that $H^2 = 1$ and class $H \leqslant 2$. This completes the proof of (3.3).

(3.4) *If H is an L support subgroup of G, then $L/C_L(H^*)$ is an r group.*

Let $H^* = H_1 > \cdots > H_t = 1$ be a G chief series for H^*. Now $\bar{H}_i = H_i/H_{i+1}$ is a G chief factor of H^*. Thus $C_L(\bar{H}_i) = L$. Let L_0 be any r' subgroup of L. Then $C_{L_0}(\bar{H}_i) = L_0$ so L_0 stabilizes the chain $H^* = H_1 > \cdots > H_t = 1$. Since L_0 is an r' group L_0 centralizes H^*. Thus $C_L(H^*)$ contains every r' subgroup of L. So $L/C_L(H^*)$ is an r group proving this lemma.

(3.5) *If H is an L support subgroup of G, then H is an L_0 support subgroup of G for some $L_0 \leqslant C_L(H^*)$ where $L_0 C_L(\hat{H}) = L$.*

Let $L_0 = C_L(H^*)$. Since $L/C_L(\hat{H})$ is an $s \neq r$ group and $L/C_L(H^*)$ is an r group, $L_0 C_L(\hat{H}) = L$. Thus L_0 acts upon H as $L_0/C_{L_0}(\hat{H})$ which is G isomorphic to $L/C_L(\hat{H})$. Clearly H is an L_0 support subgroup.

(3.6) *If H is an L support subgroup of G, then $H \leqslant L$.*

Note that $L \cap R \triangle G$. Also note that $[L, \hat{H}] = \hat{H}$ so $[L, H] \leqslant L \cap H$ is not in H^*. Thus $L \cap H = H$ and $L \geqslant H$.

(3.7) *Assume H is an L support subgroup. Assume M is a subgroup of G containing L and $N \triangle M$ satisfies $\bigcap_{x \in G} N^x = 1$. Then $H^*(N \cap H) < H$.*

Assume $H^*(N \cap H) = H$. Choose L by (3.5) so that $L = C_L(H^*)$. Then $[L, H] \triangle G$ and $[L, \hat{H}] = \hat{H}$ imply that $[L, H] = H$. But then $H = [L, H] = [L, H^*(N \cap H)] \leqslant [L, H^*]\ \ [L, N \cap H] = [L, N \cap H] \leqslant N$. Thus $H \leqslant \bigcap_{x \in G} N^x = 1$, a contradiction. This completes the proof.

(3.8) *Assume that H is an L support subgroup of G. Suppose $M \leqslant G$ contains L and $N \triangle M$ satisfies $\bigcap_{x \in G} N^x = 1$. Then there is a subgroup $H_0 \leqslant H$ with unique maximal subgroup $H_0^* = H_0 \cap H^*(N \cap H)$ and a subgroup L_0, $L \geqslant L_0 > C_L(\hat{H})$ so that H_0N/N is an L_0N/N support subgroup of M/N and $H_0N/H_0^*N \simeq H_0H^*/H^*(N \cap H_0H^*)$ as an L_0 module.*

By (3.7) $H^*(N \cap H) < H$. Choose $H_0 \leqslant H$ minimal M invariant such that $H_0 \nleqslant H^*(N \cap H)$. Then $H_0{}^* = H_0 \cap H^*(N \cap H) = H_0 \cap H \cap H^*N = H_0 \cap H^*N$ is a unique maximal M invariant subgroup of H_0. Note that $H_0/H_0{}^* = H_0/H_0 \cap H^*N \simeq H_0H^*N/H^*N$ is a section of H/H^*. Since L is fixed point free on $\hat{H} = H/H^*$, $C_L(H_0/H_0{}^*) < L$. Choose $L_0 \leqslant L$ so that $L_0/C_L(H_0/H_0{}^*)$ is an M chief factor. Let X/Y be an M chief factor of $H_0{}^*N/N$. That is, X/Y is an M chief factor of $H^*N/N \simeq H^*/H^* \cap N$. But $L/C_L(H^*)$ is an r group by (3.4) so that $L_0 = C_{L_0}(X/Y)$. Thus H_0N/N is an L_0N/N support subgroup of M/N. Now

$$\begin{aligned} H_0N/H_0{}^*N &= H_0N/(H_0 \cap H^*N)N \simeq H_0/H_0 \cap (H_0 \cap H^*N)N \\ &= H_0/H_0 \cap H^*N \simeq H_0H^*N/H^*N \simeq H_0H^*/H_0H^* \cap H^*N \\ &= H_0H^*/H^*(N \cap H_0H^*). \end{aligned}$$

Since the above string involves all L_0 isomorphisms, the proof of (3.8) is complete.

(3.9) *Assume that X/Y is a G-chief factor, $F_2(G) \geqslant X > Y \geqslant F(G)$ and Y is the unique maximal subgroup of X such that $Y \lhd G$ and $Y \geqslant F(G)$. Then there are normal subgroups $H \leqslant L \leqslant X$ so that*

(1) *H is an L support subgroup of G,*
(2) *$C_X(\hat{H}) \leqslant Y$,*
(3) *$(|H|, [X : Y]) = 1$,*
(4) *$C_X(H^*)F(G) = X$,*
(5) *$H \leqslant F(G)$, and*
(6) *$C_L(H^*) = L$.*

Among all normal subgroups of G choose M minimal normal such that $MY = X$. Now $M \cap Y$ must be the unique G-invariant maximal subgroup of M, and $X/Y \simeq M/M \cap Y$ is a G-isomorphism.

Since $M \leqslant X \leqslant F_2(G)$ we know that $M/M \cap F(G)$ is an s-group for some prime s. Therefore $O^s(M) \leqslant F(G)$ and is an s'-group. Since $M \nleqslant F(G)$, $O^s(M) \neq 1$. Choose H minimal in $O^s(M)$ such that $H \lhd G$ and $C_M(H) \leqslant M \cap Y$. Now $C_M(O^s(M)) \leqslant M \cap Y$, so such an H exists.

We prove that H has a unique maximal G-invariant subgroup H^*. Suppose H_1 and H_2 are G-invariant maximal subgroups of H, and $H_1 \neq H_2$. By minimality of H, $C_M(H_i) \nleqslant M \cap Y$. The uniqueness of $M \cap Y$ then forces $C_M(H_i) = M$. But then

$$C_M(H) = C_M(H_1H_2) = C_M(H_1) \cap C_M(H_2) = M.$$

This contradiction proves the uniqueness of H^*. At this point, since $H \leqslant F(G)$ is nilpotent, we know that H is an r-group for $r \neq s$. Therefore

$$(|H|, [X : Y]) = (r, s) = 1.$$

The minimality of H forces $C_M(H^*) = M$. Let $\hat{H} = H/H^*$. Now $C_M(\hat{H}) \leqslant M \cap Y$ since $r \neq s$. Choose $L/C_M(\hat{H})$, a G-chief factor of $M/C_M(\hat{H})$. Then $L \leqslant M = C_M(H^*)$, so that L centralizes all G-chief factors of H^*. We now have $C_L(H^*) = L$; and H is an L-support subgroup of G.

Now $MF(G)$ is a G-invariant subgroup of X not contained in Y. Therefore $MF(G) = X$. So $(M \cap Y)F(G) = Y$. Since $F(G) \leqslant C_G(\hat{H})$ we have $C_X(\hat{H}) \leqslant Y$ and $C_X(H^*)F(G) \leqslant Y$.

(3.10) *Let X/Y be a nontrivial G-chief factor such that $X \cap F(G) \leqslant Y$. Then there is a G-chief factor A/B and an L-support subgroup H of G so that*

(1) *$A/B \simeq X/Y$ are G-isomorphic*;

(2) $C_A(\hat{H}) \leqslant B$.

Choose A minimal such that $A \lhd G$ and $AY = X$. Let $B = A \cap Y$ so that $A/B \simeq X/Y$ and B is the unique maximal G-invariant subgroup of A. Observe that $A \nleqslant F(G)$ since $X \cap F(G) \leqslant Y$. Choose j minimal such that $F_j(G) \geqslant A$. Therefore, $F_{j-1}(G) \cap A \leqslant B$. If $j = 2$ then we apply (3.9) to the chief factor $AF(G)/BF(G)$ obtaining the desired support subgroup $\hat{H}$.

So we may suppose that $j > 2$. We now apply induction to $G/F(G)$ for the chief factor $(XF(G)/F(G))/(YF(G)/F(G))$. Since $X \cap F_2(G) \leqslant Y$ we have $(XF(G)/F(G)) \cap (F_2(G)/F(G)) \leqslant (YF(G)/F(G))$. That is, there is a support subgroup $\bar{K} = K/F(G)$ and a chief factor $\bar{A}_0/\bar{B}_0 \simeq X/Y$ with $C_{\bar{A}_0}(\hat{\bar{K}}) \leqslant \bar{B}_0$. Let K, K^* be the inverse image in G of $\bar{K}$ and $\bar{K}^*$. Let A_0, B_0 be the inverse image in G of $\bar{A}_0$, $\bar{B}_0$. Then

$$X/Y \simeq A_0/B_0 .$$

And $C_{A0}(K/K^*) \leqslant B_0$. Now $F_2(G) \geqslant K > K^* \geqslant F(G)$ and K^* is the unique maximal G-invariant subgroup of K containing $F(G)$. By (3.9) then there is a support subgroup H of G such that $C_K(\hat{H}) \leqslant K^*$. Suppose $C_G(\hat{H}) = C$. Then C must be trivial upon K/K^* since $[C, K] \leqslant C \cap K \leqslant K^*$. So $C \leqslant C_G(K/K^*)$ and therefore $C_{A0}(\hat{H}) \leqslant C_{A0}(K/K^*) \leqslant B_0$. This completes the proof of (3.10).

(3.11) *If X/Y is a chief factor of G then there are chief factors A/B and H/K so that*

(1) $H \leqslant F(G)$,

(2) $C_A(H/K) \leqslant B$,

(3) *$A/B \simeq X/Y$ is a G-isomorphism,*

or $F(G)Y \geqslant X$.

By (3.10) we may take H to be a support subgroup with $K = H^*$.

IV. Representation Theorems

(4.1) *Assume the following holds in this section*:

(a) *AG is a solvable group with normal subgroup G and complement A where $(|A|, |G|) = 1$.*

(b) *A is nilpotent and $\mathbf{Z}_p \wr \mathbf{Z}_p$ free for all primes p.*

(c) *$\mathbf{k}$ is an algebraically closed field of characteristic c where $c \nmid |A|$.*

(d) *V is a sum of isomorphic copies of a faithful irreducible $\mathbf{k}[AG]$ module.*

(4.2) *Assume $R \triangle AG$ is a normal r subgroup such that $Z(R) \leqslant Z(AG)$ and $R/Z(R)$ is an AG chief factor. If $C_G(R/Z(R)) < G$, $\hat{A} \triangle A$, $A^* = C_{\hat{A}}(R/Z(R))$, and $C_V(\hat{A}) = (0)$, then $C_V(A^*) = (0)$.*

Let T be a characteristic abelian subgroup of R. Then $T \triangle AG$. Now R is nonabelian and $R/Z(R)$ is an AG chief factor so $T \leqslant Z(R)$. Note $V|_{Z(R)}$ is homogeneous so $Z(R)$ is cyclic. By (2.2) R is of type e'. By (2.6) we may choose an AG invariant subgroup $R_0 \leqslant R$ so that R_0 is extra special. $Z(R_0) = R_0 \cap Z(R)$ and $R_0 Z(R) = R$. Thus $R_0/Z(R_0) \simeq R/Z(R)$ is an AG isomorphism. For proof of (4.2) we may replace R by R_0. Thus we assume R is extra special.

Now $V|_R$ is homogeneous. That is, $V|_R \simeq mU$ where U is a faithful irreducible $\mathbf{k}[R]$ module. We may extend U to an ordinary AR module U^* such that U is trivial for $C_A(R)$. (See [V, (7.2)].) Now by [12, (51.7)] we may extend U^* to a projective AG module $\hat{U}$ with factor set α where

$$\hat{U}|_{AR} = U^*,$$

and

$$\hat{U}|_R = U.$$

By [12, (53.3)] we may assume α has finite order.

We therefore get a central extension $1 \to C = \langle\alpha\rangle \to G^* \to G \to 1$ so that $\hat{U}$ is an ordinary AG^* module. Note that R is identifiable as a normal subgroup of AG^*. Now by [12, (51.7)]

$$V \simeq \hat{U} \otimes W$$

where W is an ordinary AG^* module, hence a projective AG module with factor set α^{-1} and W is trivial for R.

Note that $C_{G^*}(R/Z(R)) < G^*$ since $C_G(R/Z(R)) < G$. Now by [VI, (3.2)], with $A_0 = C_A(R) = C_A(R/Z(R))$, we have $\hat{U}|_A$ contains the regular A/A_0 module. That is,

$$\hat{U}|_A \simeq U_1 \oplus 1_{A_0}|^A.$$

But then

$$V|_A \simeq (W \otimes \hat{U})|_A \simeq W|_A \otimes U_1 \oplus W|_A \otimes (1_{A_0}|^A).$$

Set $W|_A = W_0$. Now $\hat{U}|_A = U^*|_A$ is ordinary so also W_0 is ordinary. Thus

$$\begin{aligned} V|_A &\simeq W_0 \otimes U_1 \oplus W_0 \otimes (1_{A_0}|^A) \\ &\simeq W_0 \otimes U_1 \oplus W_0|_{A_0}|^A. \end{aligned}$$

By hypothesis $C_V(\hat{A}) = (0)$ so

$$C_{W_0|_{A_0}|^A}(\hat{A}) = (0).$$

By (1.11)

$$C_{W_0|_{A_0}}(A_0 \cap \hat{A}) = (0).$$

Thus $A_0 \cap \hat{A} = A^* = C_{\hat{A}}(R/Z(R))$ is fixed point free on W_0. But then

$$\begin{aligned} V|_{A^*} &\simeq W_0|_{A^*} \otimes \hat{U}|_{A^*} \\ &\simeq (\dim U)\, W_0|_{A^*}. \end{aligned}$$

Thus $C_V(A^*) = (0)$. This proves (4.2).

(4.3) *Assume that $V|_N$ is homogeneous for every normal abelian subgroup $N \leqslant G$, $N \triangle AG$. Then for each $r \mid |F(G)|$*

(1) *$O_r(G)$ is of type e',*

(2) *if H/K is an AG chief factor of $O_r(G)$ where $KZ(O_r(G)) \ngeqslant H$, then there is an extra special r subgroup $R \leqslant O_r(G)$ such that*

(a) *$R \triangle AG$,*

(b) *$RK = H$, $R \cap K = R'$,*

(c) *$R/R' \simeq H/K$ is an AG isomorphism.*

Since N is abelian normal in AG, $\mathbf{k}$ is algebraically closed, and $V|_N$ is homogeneous, N acts as scalar multiplication on V. Since V is faithful, $N \leqslant Z(AG)$ and N is cyclic.

Let $T = O_r(G)$. If N is characteristic in T and abelian then $N \triangle AG$ so $N \leqslant T \cap Z(AG)$ is cyclic and in $Z(T)$. So by (2.2) T is of type e'. By (2.7) $T/Z(T)$ is a completely reducible AG module. Let $T_0/Z(T)$ be an irreducible component such that $T_0K = H$. Now T_0 is clearly of type e'. So by (2.6) the desired extra special subgroup $R \leqslant T_0$ exists. This proves (4.3).

(4.4) *Assume* $V|_N$ *is homogeneous for every abelian normal subgroup* N *of* AG *contained in* G. *Then, if* $A^* = C_A(G/F(G))$ *and* $C_V(\hat{A}) = (0)$, *we have*

$$C_V(A^*) = (0).$$

Let H/K be an AG chief factor of $O_r(G)$ such that $C_G(H/K) < G$. Suppose $\hat{A} \triangle A$ and $C_V(\hat{A}) = (0)$. By (4.2) and (4.3) $C_{\hat{A}}(H/K)$ has trivial centralizer on V.

Write down all AG chief factors $\bar{H}_1, ..., \bar{H}_t$ of $O_r(G)$ for all primes $r \mid |F(G)|$ such that $C_G(\bar{H}_i) < G$. Let $A_0 = A$, $A_1 = C_A(\bar{H}_1), ..., A_{i+1} = C_{A_i}(\bar{H}_{i+1})$. Now $A_i \triangle A$ for all i. If $A_0, ..., A_{i-1}$ satisfy $C_V(A_j) = (0)$ then by the preceding paragraph $C_V(A_i) = (0)$. Thus $\hat{A} = A_t$ has trivial centralizer on V and A centralizes all chief factors $\bar{H}_i$.

Let L/M be any AG chief factor of G where $M \geqslant F(G)$. Then by (3.11), $\hat{A}$ centralizes L/M. But then by (1.4) $\hat{A}$ centralizes $G/F(G)$. Now $\hat{A} \leqslant A^*$ and $C_V(\hat{A}) = (0)$ so (4.4) follows.

(4.5) *Assume* V *is not induced from any subgroup* A_0G *where* $A_0 < A$. *Suppose also that* $C_V(A) = (0)$ *and* $C_V(A_0) \neq (0)$ *for all* $A_0 < A$. *Assume* $A_1G_1 < AG$ *and* U *is a homogeneous* $\mathbf{k}[A_1G_1]$ *module such that* $U|^{AG} \simeq V$. *Then* $A_1 = A$ *and if* $G_0 = \bigcap_{x \in AG} G_1$ *then* $G = G_0C_G(A)$.

First, observe that $(U|^{A_1G})|^{AG} \simeq V$. Thus $A_1 = A$.

Second, we prove that $G_1C_G(A) = G$. We do this by induction upon $[G: G_1]$. This result is obvious if $[G: G_1] = 1$, that is, $G_1 = G$. Choose $M \geqslant G_1$ maximal A invariant in G. Let $G = G^1 > G^2 > \cdots > G^t = 1$ be an AG chief series for G. Suppose $G^{i+1} \leqslant M$ and $G^i \nleqslant M$. Now $MG^i > M \geqslant G_1$ and MG^i is A invariant so that $MG^i = G$ by the maximal choice of M. Since $G^{i+1} \leqslant M$, $M \cap G^i \triangle G^i$. Clearly AM normalizes it also. Thus $M \cap G^i < G^i$ is $AMG^i = AG$ invariant. So $M \cap G^i = G^{i+1}$. Let $H = G^i \cap K = M$, $G^i = G^{i+1}$. Then H/K is an AG chief factor such that $MH = G$ and $M \cap H = K$.

Let $W = U|^{AM}$. Then W is a homogeneous AM module. By (1.3) we may choose $x_1, ..., x_t$ as coset representatives of K in H such that $C_A(x_iK) = C_A(x_i) = A_i$. (Here we mean $C_A(x_iK)$ as the centralizer in A of $x_iK \in H/K$.)

We show that $(AM)^{x_i} \cap A = C_A(x_i)$. Certainly $C_A(x_i) \leqslant (AM)^{x_i} \cap A$. Choose $a \in (AM)^{x_i} \cap A$. Now $(a_0m)^{x_i} = a$ where $a_0 \in A$, $m \in M$. Thus a_0m has order dividing $|A|$. Since $(|A|, |M|) = 1$ and M is normal in AM, there is a $y \in M$ so that $a_0m = a_1^y$ for some $a_1 \in A$. Now $a = a_1^{yx_i}$ Thus $a_1^{-1}a = a_1^{-1}a^{yx_i} = [a_1, yx_i] \in G \cap A = 1$. So $a_1 = a$ and yx_i centralizes a. Next

$$[a, y] = a^{-1}a^y = a^{-1}a^{x_i^{-1}} = [a, x_i^{-1}] \in M \cap H = K.$$

So $[a, x_iK] = K$ and a centralizes the element x_iK of H/K. So $a \in C_A(x_iK) = C_A(x_i)$. Thus $(AM)^{x_i} \cap A = C_A(x_i)$.

Order the x_i's so that $x_1, ..., x_s$ are AMx_iA coset representatives in AG. Then the Mackey decomposition tells us that

$$V|_A \simeq W|^{AG}|_A \simeq \sum_{i=1}^{s} x_i \otimes W|_{(AM)^{x_i} \cap A}|^A$$

$$= \sum_{i=1}^{s} x_i \otimes W|_{A_i}|^A$$

$$\simeq \sum_{i=1}^{s} W|_{A_i}|^A.$$

By [VI, (2.2)] we may choose some x_i so that $C_A(x_i) = C_A(H/K) = A^*$. Now $A^* \leqslant A_i$ for every i. Further, for that i such that $A^* = A_i$, $W|_A|^A$ has trivial centralizer for A if and only if $W|_{A_i} = W|_{A^*}$ has trivial centralizer for A^*. Thus A^* has trivial centralizer on W. By the same token, (1.11) also tells us A^* has trivial centralizer on $W|_{A_i}|^A$ for each i since $A^* \triangle A$ and $A^* \leqslant A_i$ so $C_V(A^*) = (0)$. But then $A^* = A$ so all $A_i = A$. That is, $MC_G(A) = G$.

Set $M_i = x_iMx_i^{-1}$, and $W_i = x_i \otimes W$ for $i = 1, ..., t$. Now AM_i is the stabilizer in AG of W_i and $W_i|^{AG} \simeq V$. Let $\bar{M}_i = M_i/C_{M_i}(W_i)$. Now A acts the same way on each W_i so A is faithful on W_i since it is faithful on V. Further, $C_{W_i}(A_0) \neq (0)$ for every $A_0 < A$. Suppose W_i is induced from a subgroup A_0M_i. Then $W_i|^{A_0G}$ induces V so that $A_0 = A$. Finally notice that, for $Ax_iG_1x_i^{-1}$, $x_i \otimes U$ induces W_i on AM_i. So for $A\bar{M}_i$ on W_i and $A\bar{G}_i$ on $x_i \otimes U$ where $\bar{G}_i = (x_iG_1x_i^{-1}C_{M_i}(W_i))/C_{M_i}(W_i)$ we have (4.5) holding since

$$[\bar{M}_i : \bar{G}_i] = [G : G_1]/[H : K] < [G : G_1].$$

Thus $\bar{G}_iC_{\bar{M}_i}(A) = \bar{M}_i$. That is, $x_iG_1x_i^{-1}C_{M_i}(W_i)\,C_{M_i}(A) = M_i$. But $M_iC_G(A) = G$ so that

$$x_iG_1x_i^{-1}C_{M_i}(W_i)\,C_G(A) = G.$$

Now $x_iC_M(W)x_i^{-1} = C_{M_i}(W)$ so that by (1.6)

$$\left[\bigcap_i x_iG_1C_M(W)x_i^{-1}\right] C_G(A) = G.$$

But $G_1 \leqslant C_M(W)$ and $\bigcap_i x_iG_1x_i^{-1} \leqslant G_1$ so that $G_1C_G(A) = G$ by induction.

Now by (1.6) again, since $G_1C_G(A) = G$, $xG_1x^{-1}C_G(A) = G$ for all $x \in C_G(A)$. So $G_0C_G(A) = \bigcap_{x \in AG} xG_1x^{-1}C_G(A) = \bigcap_{x \in C_G(A)} xG_1x^{-1}C_G(A)$. This proves (4.5).

(4.6) *Suppose V is not induced from any proper subgroup A_0G of AG. Assume $C_V(A) = (0)$ and $A^* = C_A(G/F(G))$. Then $C_V(A^*) = (0)$.*

Proof is by induction upon $\dim V + |A|$. Choose *a* counterexample minimizing this number. Clearly we may assume V is irreducible.

(1) *$V|_G$ is irreducible.*

Certainly $V|_G \simeq mU$ for some irreducible G module U. Since $(|A|, |G|) = 1$ we may extend U [V, (7.2)] to U^* an irreducible AG module. Now by [12, (51.7)] $V \simeq W \otimes U^*$ where W is an irreducible AG/G module. Note that if W is induced from a subgroup $A_0 < A$ then V is induced from A_0G. Therefore W is a primitive module for the nilpotent group A. By [12, (52.1)] $\dim W = 1$. So $V|_G \simeq U$ is irreducible.

(2) *$C_V(A_0) \neq (0)$ for any $A_0 < A$.*

Assume false. We may choose A_0 so that A_0 is maximal in A and $C_V(A_0) = (0)$. Since A is nilpotent, A_0 is normal of prime index p in A. Now induction applies to A_0G on V by (1). Thus $A_0{}^* = C_{A_0}(G/F(G))$ satisfies $C_V(A_0{}^*) = (0)$. But $A_0{}^* \leqslant A^*$ so (2) must hold.

(3) *If $N \leqslant G$ is abelian normal in AG, then $N \leqslant Z(AG)$.*

Suppose not. Choose N minimal such that $N \triangle AG$, $N \nleqslant Z(AG)$, and N abelian. Then $V|_N = V_1 \dotplus \cdots \dotplus V_t$ where the V_i are homogeneous components and $t > 1$. We may choose V_1 so that its stabilizer in AG is A_1G_1 where $A_1 \leqslant A$ and $G_1 \leqslant G$ since A is a subgroup of AG with normal Hall complement G. We view V_1 as an A_1G_1 module so that $V_1|^{AG} \simeq V$. By (4.5), $A_1 = A$ and $G_0C_G(A) = G$ for $G_0 = \bigcap_{x \in AG} G_1$. Choose $x_1 = 1, \ldots, x_t \in C_G(A)$ as coset representatives for G_1 in G. After renumbering if necessary we obtain

$$x_iV_1 = V_i .$$

Note that as A modules

$$V_i = x_iV_1 \simeq V_1 .$$

Thus, for a subgroup $B \leqslant A$, $C_V(B) = (0)$ if and only if $C_{V_1}(B) = (0)$. So

$$C_{V_1}(A) = (0),$$

and

$$C_{V_1}(A_0) \neq (0) \qquad \text{all } A_0 < A.$$

Further, V_1 as an AG_1 module is not induced from any proper subgroup A_0G_1 of AG_1. Also observe that A is faithful on V_1. Set $\bar{G}_1 = G_1/C_G(V_1)$.

Now induction applies to $A\bar{G}_1$ on V_1. Thus $F(\bar{G}_1)\, C_{\bar{G}_1}(A) = \bar{G}_1$. Let F_1 be the inverse image in G_1 of $F(\bar{G}_1)$. Now

$$G_1 = F_1 C_{G_1}(A).$$

But $G_1 C_G(A) = G$ so that

$$F_1 C_G(A) = G.$$

Now with $F_0 = \bigcap_{x \in AG} F_1{}^x = \bigcap_{x \in C_G(A)} F_1$ by (1.6) we have

$$F_0 C_G(A) = G.$$

Note that $F_0 \triangle G$ and by (1.2) $F_0 \leqslant F(G)$. Therefore

$$F(G)\, C_G(A) = G.$$

This proves (3).

Finally, (4.6) follows from (4.4).

V. Theorem B

(5.1) *Assume the following*:

(a) *AG is a solvable group with normal subgroup G and complement A where* $(|A|, |G|) = 1$.

(b) *A is nilpotent and $\mathbf{Z}_p \wr \mathbf{Z}_p$ free for all primes p.*

(c) *$\mathbf{k}$ is a field of characteristic c where $c \nmid |A|$.*

(d) *V is a sum of isomorphic copies of a faithful irreducible $\mathbf{k}[AG]$ module.*

(e) *R is a non-trivial L support subgroup of AG where $L \leqslant G$.*

(5.2) *Assume* (5.1). *If $A = A^0 > \cdots > A^n = 1$ is a central series for A, $C_V(A^m) = (0)$, and $C_V(A^{m+1}) \neq (0)$, then there is a subgroup $D \leqslant A^m$ so that*

(a) $C_V(A^{m+1}D) = (0)$, *and*

(b) $C_{R/R^*}(D) \geqslant C_{R/R^*}(A^{m+1})$, *and*

(c) $C_{G/F(G)}(D) \geqslant C_{G/F(G)}(A^{m+1})$.

The proof is by induction upon $\delta(AG, R, V) = \dim V + |A| + |G|$. We assume (5.2) is false and choose a counterexample minimizing $\delta(AG, R, V)$. For (AG, R, V) we may assume that for any $D < A^m$, at least one of the following occurs:

(5.3) (1) $C_V(A^{m+1}D) \neq (0)$, or

(2) $C_{R/R^*}(D) \ngeqslant C_{R/R^*}(A^{m+1})$, or

(3) $C_{G/F(G)}(D) \ngeqslant C_{G/F(G)}(A^{m+1})$.

The first step is quite usual.

(5.4) *V is irreducible.*

Suppose $V = V_1 \dotplus \cdots \dotplus V_t$ where the V_i are irreducible. Then (5.2) holds for (AG, R, V_i) if and only if it holds for (AG, R, V). Thus $t = 1$.

(5.5) *We may assume* $\mathbf{k}$ *is algebraically closed.*

Let $\hat{\mathbf{k}}$ be an algebraic closure of $\mathbf{k}$ and $\hat{\mathbf{k}} \otimes_{\mathbf{k}} V = \hat{V}$. Now

$$\hat{V} = \hat{V}_1 \dotplus \cdots \dotplus \hat{V}_t$$

where the $\hat{V}_i$ are homogeneous components, all being algebraically conjugate. By (1.10) we see that if (5.2) holds for (AG, R, V_i), it holds for (AG, R, V). Thus we may assume $t = 1$ and $\hat{\mathbf{k}} = \mathbf{k}$.

(5.6) *V is not induced from any proper subgroup A_0G of AG. In particular, $V\,|_{A_0G}$ is homogeneous for every $A_0 \triangle A$.*

The second sentence is clearly a consequence of the first. So assume V is induced from a subgroup A_0G where A_0 is a proper subgroup of A. Since A is nilpotent, we may assume that A_0 is maximal (hence normal of prime index p) in A. Now there is an irreducible $\mathbf{k}[A_0G]$ module U so that $U\,|^{AG} \simeq V$.

With $M = A_0G$, $N = \ker[A_0G \to \operatorname{Aut} U]$, and $H = R$ we may apply (3.8) to obtain $L_0 \leqslant L$ and $H_0 \leqslant R$ so that H_0N/N is an L_0N/N support subgroup of A_0G/N. Further, the unique maximal A_0G/N invariant subgroup of H_0N/N is $H_0{}^*N/N$ where

$$H_0{}^* = H_0 \cap R^*(N \cap R).$$

Let $\overline{A_0G} = A_0G/N$, $\bar{H}_0 = H_0N/N$, and $L_0 = L_0N/N$.

By (1.11) $C_U(A_0 \cap A^k) = (0)$ if and only if $C_V(A^k) = (0)$. So with $A_0{}^k = A_0 \cap A^k$ we have $C_U(A_0{}^m) = (0)$ and $C_U(A_0^{m+1}) \neq (0)$. We now apply induction to $(\overline{A_0G}, \bar{H}_0, U)$. Thus there is $D \leqslant A_0{}^m$ so that

(a) $C_U(A_0^{m+1}D) = (0)$, and

(b) $C_{H_0/H_0{}^*}(D) \geqslant C_{H_0/H_0{}^*}(A_0^{m+1})$, and

(c) $C_{\bar{G}/F(\bar{G})}(D) \geqslant C_{\bar{G}/F(\bar{G})}(A_0^{m+1})$.

Notice that $A_0{}^k = A_0 \cap A^k$ is normal in A and $A_0{}^k/A_0^{k+1}$ is central. So $A_0^{m+1}D$ is normal in A. Choose $1 = x_1, \ldots, x_p \in A$ coset representatives for A_0G in AG. Then, with $U_i = x_i \otimes U$, we have $C_{U_i}(A_0^{m+1}D) = C_{U_i}(x_iA_0^{m+1}Dx_i^{-1}) = (0)$. Thus

(a) $C_V(A^{m+1}D) \leqslant C_V(A_0^{m+1}D) = (0)$.

Note that $\hat{R} = R/R^* = \hat{R}_1 \dotplus \cdots \dotplus \hat{R}_t$ as an A_0G module where the $\hat{R}_i$ are homogeneous components and $t = 1$ or p. We may number the $\hat{R}_i$ so that $H_0/H_0^* \simeq H_0R^*/R^*$ is an irreducible component of $\hat{R}_1$. (See (3.8).) If $t = p$ we may also set $\hat{R}_i = x_i\hat{R}_1$.

Since $\hat{R}_1$ is homogeneous, by (b) above, we have

$$C_{\hat{R}_1}(D) \geqslant C_{\hat{R}_1}(A_0^{m+1}).$$

Suppose for a moment $t = p$. Then, since $A^{m+1}D$ is normal in A,

$$\begin{aligned} C_{\hat{R}_i}(A_0^{m+1}D) &= C_{\hat{R}_i}(x_iA_0^{m+1}Dx_i^{-1}) \\ &\geqslant C_{\hat{R}_i}(x_iA_0^{m+1}x_i^{-1}) \\ &\geqslant C_{\hat{R}_i}(A_0^{m+1}). \end{aligned}$$

Thus $C_{\hat{R}}(A_0^{m+1}D) \geqslant C_{\hat{R}}(A_0^{m+1})$ whether $t = 1$ or p. So

$$\begin{aligned} \text{(b)} \quad C_{\hat{R}}(D) &\geqslant C_{\hat{R}}(A_0^{m+1}D) \\ &\geqslant C_{\hat{R}}(A_0^{m+1}) \\ &\geqslant C_{\hat{R}}(A^{m+1}). \end{aligned}$$

Finally we work on (c). Set $\bar{G}_i = Gx_iNx_i^{-1}/x_iNx_i^{-1}$. Then $\bar{A}_0\bar{G}_i$ acts upon $V_i = x_i \otimes U$. From (c) above we obtain

$$C_{\bar{G}_i/F(\bar{G}_i)}(A_0^{m+1}D) \geqslant C_{\bar{G}_i/F(G_i)}(A_0^{m+1})$$

just as above for R_i. But now, since V is faithful, $\bigcap x_iNx_i^{-1} = 1$ and by (1.8)

$$\begin{aligned} \text{(c)} \quad C_{G/F(G)}(D) &\geqslant C_{G/F(G)}(A_0^{m+1}D) \\ &\geqslant C_{G/F(G)}(A_0^{m+1}) \\ &\geqslant C_{G/F(G)}(A^{m+1}). \end{aligned}$$

This contradiction proves (5.6).

(5.7) *For every* $A_0 < A$, $C_V(A_0) \neq (0)$. *In particular*, $M = 0$.

Clearly the first part implies the second. So we prove the first part. Assume it is false. Let A_0 be any subgroup of A such that $[A\colon A_0]$ is a prime p and $C_V(A_0) = (0)$. Certainly one such subgroup exists by our assumption and the nilpotence of A.

Now $V|_{A_0G}$ is homogeneous by (5.6) and also $\hat{R}|_{A_0G} = \hat{R}_1 \dotplus \cdots \dotplus \hat{R}_t$ where each $\hat{R}_i$ is homogeneous and $t = 1$ or p. By (3.8) again we may choose $H_0 \leqslant R$ an $L_0 \leqslant L$ support subgroup of A_0G so that $H_0{}^* = H_0 \cap R^*$ and H_0R^*/R^* is an irreducible component of $\hat{R}_1$. So induction again applies to (A_0G, H_0, V). Suppose $A_0{}^k = A_0 \cap A^k$ and $C_V(A_0{}^k) = (0)$ but $C_V(A_0^{k+1}) \neq (0)$. Then there is a subgroup $D \leqslant A_0{}^k$ so that

(a) $C_V(A_0^{k+1}D) = (0)$, and

(b) $C_{H_0}(D) \geqslant C_{H_0}(A_0^{k+1})$, and

(c) $C_{G/F(G)}(D) \geqslant C_{G/F(G)}(A_0^{k+1})$.

Just as in (5.6) these lead to

(a) $C_V(A^{k+1}D) = (0)$, and

(b) $C_{\hat{R}}(D) \geqslant C_R(A^{k+1}D)$, and

(c) $C_{G/F(G)}(D) \geqslant C_{G/F(G)}(A^{k+1}D)$.

We will have proved (5.7) if we can show that $k = m$.

Note that the above argument works for any A_0 of prime index p in A such that $C_V(A_0) = (0)$. First, assume $m > 0$. Then we may choose $A_0 \geqslant A^1$. Thus $A_0{}^j = A^j$ for all $j > 0$. So $k = m$. Second, we assume $m = 0$. Then $C_V(A^j) \neq (0)$ for any $j > 0$. Thus $C_V(A_0{}^j) \neq (0)$ for any $j > 0$ and $k = 0$. So again $k = m = 0$. This completes (5.7).

(5.8) *$C_A(\hat{R}) < A$ or $C_A(G/F(G)) < A$.*

Assume that A centralizes $\hat{R}$ and $G/F(G)$. Then with $D = A$ (5.2) holds since $m = 0$.

(5.9) *$C_A(G/F(G)) = A$ and $C_A(\hat{R}) < A$.*

We have V not induced from any proper subgroup A_0G of AG by (5.6). So by (4.6) $A^* = C_A(G/F(G))$ satisfies

$$C_V(A^*) = (0).$$

By (5.7) we must have $A^* = A$. The rest follows by (5.8).

We now endeavor to prove that $R^* \leqslant Z(AG)$. The proof is rather involved. We make the following:

(5.10) Assumption. $R^* \nleqslant Z(AG)$.

Among all subgroups of R^* we choose Q minimal normal in AG such that $Q \nleqslant Z(AG)$. That is, $Q/Z(AG) \cap Q$ is an AG chief factor.

(5.11) *Q is abelian.*

Suppose not. Let N be a characteristic abelian subgroup of Q. Now $N \triangle AG$ so that by the minimal choice of Q, $N \leqslant Z(AG) \cap Q \leqslant Z(Q)$. So by (2.2) Q is of type e'. By (2.6) and the minimality of Q we see that Q is extra special. Now $Q/Z(Q)$ is an AG chief factor so that R is trivial on $Q/Z(Q)$. So by (1.9) $R = C_R(Q)Q$. Both $C_R(Q)$, Q are proper AG invariant subgroups of R so that $R = C_R(Q)Q \leqslant R^*$. This contradiction proves (5.11).

Another way is to note we may take $L \leqslant C_G(R^*)$ by (3.5). Then by (3.6) $Q \leqslant R \leqslant L \leqslant C_G(R^*) \leqslant C_G(Q)$.

Since $Q \nleqslant Z(AG)$ and Q is abelian we know that $V|_Q = V_1 \dotplus \cdots \dotplus V_t$ where the V_i are homogeneous components and $t > 1$. Let S be the stabilizer in AG of V_1. If $xV_1 = V_i$ then xSx^{-1} is the stabilizer in AG of V_i. Now A is a Hall subgroup of AG and G is a normal Hall complement to A so that we may number the V_i such that

$$S = A_1G_1$$

where $A_1 = A \cap S$ and $G_1 = G \cap S$. Let $G_0 = \bigcap_{x\in AG} G_1$. By (5.6) and (5.7) we may apply (4.5). Thus:

(5.12) *The stabilizer of V_1 is AG_1 where $G_1 \leqslant G$. If $G_0 = \bigcap_{x\in AG} G_1$, then $G_0C_G(A) = G$. Further, A centralizes Q.*

Choose $x_1 = 1, x_2, \ldots, x_t \in C_G(A)$ so that $x_iV_1 = V_i$. Now Q is abelian so that Q acts as scalar multiplication upon V_i. In particular, A centralizes the action of Q on V_1. Thus $A = x_iAx_i^{-1}$ centralizes the action of $Q = x_iQx_i^{-1}$ upon $V_i = x_iV_1$. We conclude that A centralizes the action of Q on V. Since V is faithful, this means that A centralizes Q.

Remark. By (3.5) we may assume that $L \leqslant C_G(R^*)$. Thus $R \leqslant L \leqslant C_G(Q)$ by (3.6).

(5.13) *There is a normal subgroup $M < G$ of prime index p such that $AM \triangle AG$ and $AM > C_G(Q)$.*

Certainly $C_{AG}(Q) \geqslant AL$ is normal in AG. Choose $K \geqslant C_{AG}(Q)$ maximal normal in AG. Then with $M = K \cap G$ we have (5.13).

(5.14) $R^* \leqslant Z(AG)$. *Assumption* (5.10) *is false.*

Suppose $V|_{AM} = U_1 \dotplus \cdots \dotplus U_s$ where the U_i are the homogeneous components and $s = 1$ or p. Now $MC_G(A) = G$ so we choose $x_1 = 1, \ldots, x_p \in C_G(A)$ as coset representatives of AM in AG. Let $N = \ker[AM \to \operatorname{Aut} U_1]$. By (3.8) we may choose a subgroup $H_0 \leqslant R$ and $L_0 \leqslant L$ so that H_0N/N is an L_0N/N support subgroup of AM/N.

Now $\hat{R}|_{AM} = \hat{R}_1 + \cdots + \hat{R}_m$ where the $\hat{R}_i$ are homogeneous components and $m = 1$ or p. We number the $\hat{R}_i$ so that

$$\hat{H}_0 \simeq H_0R^*/R^* \leqslant \hat{R}_1 .$$

Since $x_i \in C_G(A)$, each $U_i = x_iU_1$ is isomorphic to U_1 as an A module. So a subgroup $B \leqslant A$ satisfies $C_{U_1}(B) = (0)$ if and only if $C_V(B) = (0)$.

We may now apply induction to $(AM/N, H_0N/N, U_1)$. Because of (5.7) we obtain

(a) $C_V(A) = (0)$,

(b) $C_{\hat{H}_0}(A) \geqslant C_{\hat{H}_0}(A^1)$.

But $\hat{R}_1$ is homogeneous and $\hat{H}_0$ is AM isomorphic to an irreducible component so that

$$C_{\hat{R}_1}(A) \geqslant C_{\hat{R}_1}(A^1).$$

But x_i centralizes A so if $\hat{R}_i = x_i\hat{R}_1$,

$$C_{\hat{R}_i}(A) \geqslant C_{\hat{R}_i}(A^1).$$

Applying (5.9) we obtain with $D = A$,

(a) $C_V(A) = (0)$,

(b) $C_{\hat{R}}(A) \geqslant C_{\hat{R}}(A^1)$,

(c) $C_{G/F(G)}(A) \geqslant C_{G/F(G)}(A^1)$.

This proves (5.14).

(5.15) *R is abelian.*

Suppose R is non-abelian. If $N \leqslant R$ is characteristic and abelian then $N \triangle AG$ so $N \leqslant R^* \leqslant Z(AG)$. Thus R is of type e' by (2.2). But by (2.6) we may assume that R is extra special. Now by (4.2) with $A^* = C_A(R)$, we have $C_V(A^*) = (0)$. By (5.7), $A^* = A$. But by (5.9) $A^* < A$. This proves (5.15).

(5.16) (5.2) *holds.*

Consider $V|_R = V_1 \dotplus \cdots \dotplus V_t$ where the V_i are homogeneous components. As in (5.4) we may number the V_i so that the stabilizer of V_1 is AG_1 for $G_1 \leqslant G$ and $G_1C_G(A) = G$. Now the argument following (5.12) shows that A centralizes R. But then

$$C_V(A) = (0) \qquad \text{and} \qquad C_V(A^1) \neq (0)$$

and

(a) $C_V(A) = (0)$,

(b) $C_R(A) \geqslant C_R(A^1)$,

(c) $C_{G/F(G)}(A) \geqslant C_{G/F(G)}(A^1)$.

This final contradiction proves (5.16) and therefore (5.2).

Remark. In the hypotheses, $A = A^0 > \cdots > A^n = 1$ is a strictly decreasing central series. Often we took, for $A_0 < A$, the central series given by $A_0 \cap A^k$. This need not strictly decrease. But by dropping out repetitions it will. So, the fact that we ignored this is of no great consequence in the proof. The reason for the strict decrease was to ensure that $A^m = A$ if and only if $m = 0$. Actually, all we needed was the central section A^m/A^{m+1}.

VI. Theorem A

(6.1) *Assume the following*:

(a) *AG is a solvable group with normal subgroup G and complement A where* $(|A|, |G|) = 1$.

(b) *A is nilpotent and $\mathbf{Z}_p \wr \mathbf{Z}_p$ free for all primes p.*

(c) *$\mathbf{k}$ is a field of characteristic c where $c \nmid |A|$.*

(d) *V is a sum of isomorphic copies of a faithful irreducible $\mathbf{k}[AG]$ module.*

(e) *$R \leqslant G$ is abelian normal in AG.*

(6.2) *If $A = A^0 > A^1 > \cdots > A^n = 1$ is a central series for A, $C_V(A^m) = (0)$, and $C_V(A^{m+1}) \neq (0)$, then there is a subgroup $D < A^m$ so that*

(1) $C_V(A^{m+1}D) = (0)$, *and*

(2) $C_R(D) \geqslant C_R(A^{m+1})$.

The proof is by induction upon $\dim V + |A| + |G|$. We assume (6.2) is false and choose a minimal counterexample (AG, V).

(1) *V is irreducible.*

This is as (5.3).

(2) *We may assume $\mathbf{k}$ is algebraically closed.*

Again we follow (5.4).

(3) $R \nleqslant C_G(A)$; $R \nleqslant Z(AG)$.

If $R \leqslant C_G(A)$ then (1) and (2) of (6.2) hold for $D = A^m$. If $R \nleqslant C_G(A)$ then $R \nleqslant Z(AG)$.

We write $V|_R = V_1 \dot{+} \cdots \dot{+} V_t$ where the V_i are homogeneous components. By (3) $t > 1$. Since A has normal Hall complement G, by taking an appropriate conjugate we may assume that the stabilizer in AG of V_1 is A_1G_1 where $A_1 \leqslant A$, $G_1 \leqslant G$. Then as an A_1G_1 module we have $V_1(A_1G_1)|^{AG} \simeq V$.

(4) *V is not induced from any proper subgroup A_0G of AG. $A_1 = A$.*

Suppose not. Choose A_0 maximal (hence normal of prime index p) in A so that there is an A_0G module U so that $U|^{AG} \simeq V$. Let

$$N = \ker[A_0G \to \operatorname{Aut} U].$$

Set $\overline{A_0G} = A_0G/N$, $\bar{R} = RN/N$. By (1.11) we know that $C_U(A_0) = (0)$. Thus induction applies to (A_0G, U). Let $A_0{}^k = A_0 \cap A^k$. Then again, by (1.11), $C_U(A_0{}^m) = (0)$ and $C_U(A_0^{m+1}) \neq (0)$. So there is a $D \leqslant A_0{}^m$ with

(a) $C_U(A_0^{m+1}D) = (0)$, and

(b) $C_{\bar{R}}(D) \geqslant C_{\bar{R}}(A_0^{m+1})$.

Choose $x_1 = 1,\ldots, x_p \in A$ as coset representatives for A_0 in A. Since $A_0^{m+1}D$ is normal in A, as in (5.6), we find that

$$C_V(A_0^{m+1}D) = (0).$$

Also with $N_i = x_iNx_i^{-1}$ and $\bar{R}_i = RN_i/N_i$ we get, as in (5.6),

$$C_{\bar{R}_i}(A_0^{m+1}D) \geqslant C_{\bar{R}_i}(A_0^{m+1})$$

for all i.

But now by (1.3)

$$C_R(A_0^{m+1}D)N_i \geqslant C_R(A_0^{m+1})N_i$$

for all i. So by (1.7), since $\bigcap N_i = 1$,

$$C_R(A_0^{m+1}D) \geqslant C_R(A_0^{m+1}).$$

Mimicking (5.6) still we then obtain

(a) $C_V(A^{m+1}D) = (0)$, and

(b) $C_R(D) \geqslant C_R(A^{m+1})$.

Since we may choose $A_0 \geqslant A_1$ and $U = V_1(A_1G_1)|^{A_0G}$ if $A_1 < A$ this completes (4).

(5) $C_V(A_0) \neq (0)$ *for any* $A_0 < A$, $m = 0$.

Suppose not. Choose A_0 maximal in A such that $C_V(A_0) = (0)$. Now induction applies to (A_0G, V). If we set $A_0{}^k = A_0 \cap A^k$ then, for some k, $C_V(A_0{}^k) = (0)$ and $C_V(A_0^{k+1}) \neq (0)$. There is a $D \leqslant A_0{}^k$ so that

(a) $C_V(A_0^{k+1}D) = (0)$, and

(b) $C_R(D) \geqslant C_R(A_0^{k+1})$.

If $m > 1$ then we may choose $A_0 \geqslant A^1$ and $k = m$. If $m = 0$ then $C_V(A^j) \neq (0)$ all $j > 0$ so $C_V(A_0{}^j) \neq (0)$ for all $j > 0$. So again $k = 0 = m$. So (5) follows much as (5.7) did.

(6) $G_1C_G(A) = G$.

This is an immediate consequence of (4.5).

(7) *A centralizes R.*

Note that we may choose $x_1 = 1, x_2, \ldots, x_t \in C_G(A)$ as coset representatives for G_1 in G. Now we may number things so that $x_iV_1 = V_i$. Then as A modules all V_i are isomorphic to V_1. Further, R acts as scalar multiplication on V_1, hence on each V_i. Thus A centralizes the action of R on V. Since V is faithful, A centralizes R. Recall (5.12).

(8) (6.2) *holds.*

We compare (7) and (3).

References

1. S. F. Bauman, The Klein group as an automorphism group without fixed points, *Pacific J. Math.* **18** (1966), 9–13.
2. T. R. Berger, Class two p groups as fixed point free automorphism groups, *Illinois J. Math.* **14** (1970), 121–149.
3. T. R. Berger, Hall-Higman type theorems. I, to appear.
4. T. R. Berger, Hall-Higman type theorems. II, to appear.
5. T. R. Berger, Hall-Higman type theorems. III, to appear.
6. T. R. Berger, Hall-Higman type theorems. IV, to appear.
7. T. R. Berger, Hall-Higman type theorems. V, to appear.
8. T. R. Berger, Hall-Higman type theorems. VI, to appear.
9. T. R. Berger, Nilpotent fixed point free automorphism groups of solvable groups, to appear.
10. T. R. Berger, Nilpotent groups as fixed point free automorphism groups, *Amer. Math. Soc. Notices*, Abstract 663–95 p. 99.
11. T. R. Berger, Odd p groups as fixed point free automorphism groups, *Illinois J. Math.* **15** (1971), 28–36.
12. C. W. Curtis and I. N. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Interscience, New York, 1962.

13. George Glauberman, Fixed points in groups with operator groups, *Math. Z.* **84** (1964), 120–125.
14. D. Gorenstein, "Finite Groups," Harper & Row, New York, 1968.
15. P. Hall and G. Higman, The p-length of a p-solvable group and reduction theorems for Burnside's problem, *Proc. London Math. Soc.* (3) **6** (1956), 1–42.
16. F. Hoffman, Nilpotent height of finite groups admitting fixed point free automorphisms, *Math. Z.* **85** (1964), 260–267.
17. E. Shult, On groups admitting fixed point free abelian operator groups, *Illinois J. Math.* **9** (1965), 701–720.